

# Quasisymmetric geometry of the carpet Julia sets

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## Abstract

Let  $J_f$  be a Sierpiński carpet which is the Julia set of rational map  $f$  and  $\mathcal{C}$  the set of all peripheral circles of this carpet. We prove that  $J_f$  is quasisymmetrically equivalent to a round carpet if the elements in  $\mathcal{C}$  avoid the  $\omega$ -limit sets of all critical points of  $f$ . Suppose that  $f$  is semi-hyperbolic, then the elements in  $\mathcal{C}$  are uniform quasicircles. Moreover, the elements in  $\mathcal{C}$  are uniformly relatively separated if and only if they are disjoint with the  $\omega$ -limit sets of all critical points.

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## 1 Introduction

Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces. If there exist a homeomorphism  $f : X \rightarrow Y$  and a distortion control function  $\eta : [0, \infty) \rightarrow [0, \infty)$  which is also a homeomorphism such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta \left( \frac{d_X(x, y)}{d_X(x, z)} \right)$$

for every distinct points  $x, y, z \in X$ , then  $f$  is called a *quasisymmetric map* and  $(X, d_X)$ ,  $(Y, d_Y)$  are called *quasisymmetrically equivalent* to each other. A basic question in quasiconformal geometry is to determine whether two given homeomorphic spaces are quasisymmetrically equivalent to each other.

It is known that the question arises also in the classification of hyperbolic spaces and word hyperbolic groups in the sense of Gromov [BP, Kl]. See also [Bou] for examples of inequivalent spaces modelled on the universal Menger curve. In this paper, we focus our attention on the Sierpiński carpets that arise as the Julia sets of rational maps.

According to [Wh], a set  $S \subset \widehat{\mathbb{C}}$  is called a *Sierpiński carpet* (*carpet* in short) if  $S$  has empty interior and can be expressed as  $S = \widehat{\mathbb{C}} \setminus \bigcup_{i \in \mathbb{N}} D_i$ , where  $\{D_i\}$  are pairwise disjoint Jordan disks with  $\text{diam}(D_i) \rightarrow 0$  as  $i \rightarrow \infty$ . The collection of the boundaries of the Jordan disk  $\{\partial D_i\}_{i \in \mathbb{N}}$  are called the *peripheral circles* of  $S$ . If each peripheral circle  $\partial D_i$  is a round circle, then  $S$  is called a *round carpet*. All Sierpiński carpets are homeomorphic to each other, so the question about the quasimetric classification of the Sierpiński carpets arises naturally.

Actually, the study of the quasimetric equivalences between the Sierpiński carpets and round carpets was partially motivated by the Kapovich-Kleiner conjecture in the geometry group theory. This conjecture is equivalent to the following statement: if the boundary of infinity  $\partial_\infty G$  of a Gromov hyperbolic group  $G$  is a Sierpiński carpet, then  $\partial_\infty G$  is quasimetrically equivalent to a round carpet in  $\widehat{\mathbb{C}}$ .

As the Julia set of a rational map, the first example of Sierpiński carpet was found by Tan [Mil, Appendix F]. Later, the rational maps whose Julia sets are Sierpiński carpets appeared in many literatures. Such as the McMullen maps [DLU], the generated McMullen maps [XQY] and the quadratic rational maps [DFGJ] etc.

Let  $f$  be a rational map whose Julia set  $J_f$  is a Sierpiński carpet. Two questions arise naturally: (Q1) Can one give another rational map  $g$  whose Julia set  $J_g$  is a Sierpiński carpet, but  $J_g$  is not quasimetrically equivalent to  $J_f$ ? This question is equivalent to ask whether there exist quasimetrically inequivalent carpet Julia sets. (Q2) Can  $J_f$  be quasimetrically equivalent to a round carpet?

Let  $X$  be a metric space. The *conformal dimension* of  $X$  is the infimum of the Hausdorff dimensions of all metric spaces which are quasimetrically equivalent to  $X$ . By definition, it is easy to see the conformal dimension is invariant under the quasimetric maps. For the first question stated above, Haïssinsky and Pilgrim constructed a sequence of hyperbolic rational maps with carpet Julia sets and showed that their conformal dimensions tend to two [HP, Theorem 3]. This means that there are infinitely many quasimetrically inequivalent Sierpiński carpets as the Julia sets of rational maps.

The *relative distance*  $\Delta(A, B)$  of two sets  $A$  and  $B$  in  $\widehat{\mathbb{C}}$  is defined as

$$\Delta(A, B) := \frac{\text{dist}(A, B)}{\min\{\text{diam}(A), \text{diam}(B)\}}, \quad (1.1)$$

where  $\text{dist}(A, B) := \sup_{a \in A, b \in B} |a - b|$  is the *distance* between  $A$  and  $B$ , and  $\text{diam}(A) := \sup_{a_1, a_2 \in A} |a_1 - a_2|$  is the *diameter* of  $A$ . A set of Jordan curves  $\mathcal{C} = \{C_i\}_{i \in \mathbb{N}}$  is called *uniformly relatively separated* if their pairwise relative distances are uniformly bounded away from zero. Specifically, there exists  $\delta > 0$  such that  $\Delta(C_i, C_j) \geq \delta$  for every two different  $i$  and  $j$ . The set  $\mathcal{C}$  are *uniform quasicircles* if there exists  $K \geq 1$  such that each  $C_i$  in  $\mathcal{C}$  is a  $K$ -quasicircle.

For the question (Q2), Bonk gave a sufficient condition on the carpets in  $\widehat{\mathbb{C}}$  such that they can quasimetrically equivalent to some round carpets. He proved that a carpet  $S$  in  $\widehat{\mathbb{C}}$  is quasimetrically equivalent to a round carpet if its peripheral circles are uniform quasicircles and is uniformly relatively separated [Bon, Corollary 1.2]. It is worth to mention that quasimetric maps preserve the uniform quasicircles and uniformly relatively separated properties. It is not hard to see that the peripheral circles of such  $S$  must be uniform quasicircles but are not necessarily uniformly relatively separated.

Recently, Bonk, Lyubich and Merenkov studied the postcritically-finite rational maps whose Julia sets are Sierpiński carpets. They proved that if the Julia set of a sub-hyperbolic rational map is a Sierpiński carpet, then it is quasimetrically equivalent to a round carpet [BLM, Theorem 1.10]. They also consider the quasimetric group between the carpet Julia sets of postcritically-finite rational maps and proved that any quasimetric map  $\xi$  defined from a carpet  $J_f$  onto a carpet  $J_g$  must be the restriction of a Möbius transformation, where  $f$  and  $g$

are postcritically-finite rational maps [BLM, Theorem 1.1]. As a corollary, they proved that the group  $QS(J_f)$ , consisting of quasisymmetric self-map of  $J_f$ , is finite [BLM, Corollary 1.2].

In this paper, we study carpet Julia sets in postcritically-infinite case.

**1.1. Statement of the main results.** The  $\omega$ -limit set  $w(x)$  of a point  $x \in \widehat{\mathbb{C}}$  under a rational map  $f$  is defined as the set of accumulation points in the orbit of  $x$ . More precisely,  $w(x) := \{y \in \widehat{\mathbb{C}} : \text{there exists a sequence } \{k_n\}_{n \in \mathbb{N}} \text{ such that } \lim_{n \rightarrow \infty} f^{\circ k_n}(x) = y\}$ . Obviously,  $w(x)$  is  $f$ -forward invariant. We establish a sufficient condition on the carpet Julia sets such that they are quasisymmetrically equivalent to some round carpets.

**Theorem 1.1.** *Let  $f$  be a rational map whose Julia set  $J_f$  is a Sierpiński carpet. If the boundaries of the periodic Fatou components are disjoint with the  $\omega$ -limit sets of the critical points, then the peripheral circles of  $J_f$  are uniform quasicircles and uniformly relatively separated. In particular,  $J_f$  is quasisymmetrically equivalent to a round carpet.*

Recall that a rational map is *sub-hyperbolic* if every critical orbit is either finite or converges to an attracting periodic orbit. Note that the boundary of each Fatou component cannot contain any critical point if the Julia set is a Sierpiński carpet. By Theorem 1.1, we have following immediate corollary.

**Corollary 1.2.** *Let  $f$  be a sub-hyperbolic rational map whose Julia set  $J_f$  is a Sierpiński carpet. Then the peripheral circles of  $J_f$  are uniform quasicircles and uniformly relatively separated. In particular,  $J_f$  is quasisymmetrically equivalent to a round carpet.*

A critical point  $c$  of  $f$  is called *recurrent* if  $c \in w(c)$ . A rational map  $f$  is called *semi-hyperbolic* if and only if the Julia set  $J_f$  contains neither parabolic periodic points nor recurrent critical points (see [Ma] and [Yin]). It was known that the Julia set of a semi-hyperbolic rational map is locally connected and has measure zero or equal to  $\widehat{\mathbb{C}}$ .

**Theorem 1.3.** *Let  $f$  be a semi-hyperbolic rational map whose Julia set  $J_f$  is a Sierpiński carpet. Then the peripheral circles of  $J_f$  are uniform quasicircles. Moreover, they are uniformly relatively separated if and only if the  $\omega$ -limit sets of the critical points are disjoint with the boundaries of periodic Fatou components.*

If a rational map is not semi-hyperbolic, then the boundary of some Fatou component may not be a quasicircle although it is a Jordan curve. For example, one can construct a rational map  $f$  whose Julia set is a Sierpiński carpet but the Julia set  $J_f$  contains a parabolic periodic point. The corresponding parabolic Fatou component contains exactly one petal and has infinitely many cusps on its boundary. Thus the boundary of this Fatou component cannot be a quasicircle. In this case,  $J_f$  cannot quasisymmetrically equivalent to a round carpet. See Figure 1.

As a corollary, we have the following theorem.

**Theorem 1.4.** *Let  $f$  be a semi-hyperbolic rational map whose Julia set  $J_f$  is a Sierpiński carpet. Then the quasisymmetric group  $QS(J_f)$  is discrete.*

**1.2. Outline of the proof and the organization of this paper.** We are mainly interested on the condition when a carpet Julia set is quasisymmetrically equivalent to a round carpet. By Bonk's criterion, this motivates us to find the condition when the peripheral circles of a carpet Julia set are uniform quasicircles and when they are uniformly relatively separated.

In order to prove the peripheral circles of some carpet Julia sets are uniform quasicircles, we first discuss the periodic Fatou components and prove that they are quasicircles if their

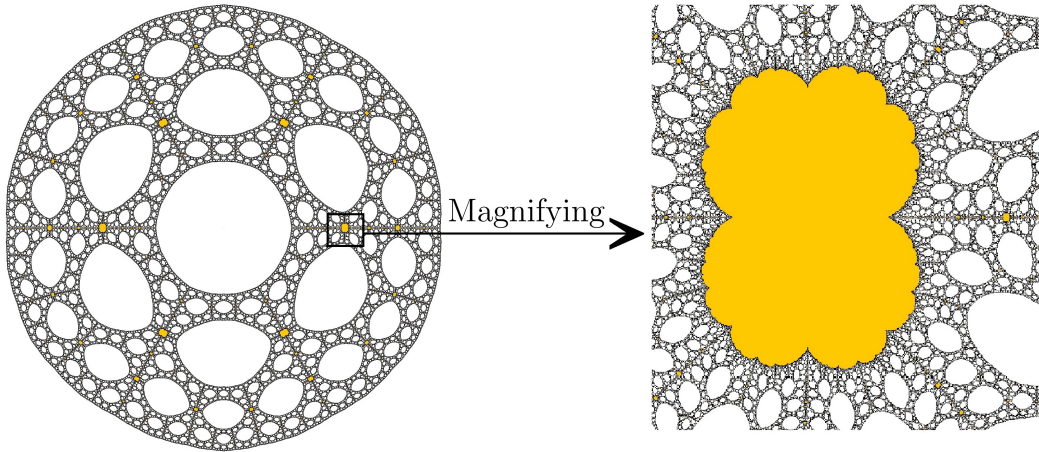


Figure 1: The Julia set of  $f(z) = z^3 + \lambda/z^3$  and an enlargement of a parabolic Fatou component, where  $\lambda \approx 0.02772313$  such that  $J_f$  is a Sierpiński carpet containing a parabolic periodic point. The peripheral circles of  $J_f$  are not uniform quasicircles but they are uniformly relatively separated.

boundaries avoid the parabolic periodic points and the points in the  $\omega$ -limit sets of the recurrent critical points (Lemma 3.5). Therefore, all peripheral circles are quasicircles by using Sullivan's eventually periodic theorem. In order to prove the uniformity, we discuss two cases. The first case, suppose that all the periodic Fatou components are disjoint with the  $\omega$ -limit sets of the critical points. Then for each periodic Fatou component  $U$ , one can find a large Jordan disk  $V$  such that  $V \setminus \overline{U}$  is an annulus and all components of the preimages of  $V \setminus \overline{U}$  are annuli whose moduli have uniform lower bound. By using a distortion argument, one can prove that all peripheral circles are uniform quasicircles (Proposition 3.6). The second case, suppose that the rational map is semi-hyperbolic. Then the corresponding Julia set (and hence all the periodic Fatou components) contains neither parabolic periodic points nor recurrent critical points. One can also prove that all peripheral circles are uniform quasicircles by using Mañé's theorem and its variation (Theorem 3.1, Lemma 3.2 and Proposition 3.7).

In order to prove the peripheral circles of some carpet Julia sets are uniformly relatively separated, we first establish a lemma which asserts that the modulus can control the relative distance (Lemma 2.4). Then we prove the peripheral circles are uniformly relatively separated by showing that all moduli of the annuli between two different peripheral circles have a lower positive bound (Proposition 3.8).

This paper is organized as follows: In §2, we prepare some distortion lemmas for the proofs of Theorems 1.1 and 1.3. Moreover, we prove that the modulus can control the relative distance. In §3, we first prove some propositions about the properties of uniform quasicircles and uniformly relatively separated. Then we prove Theorem 1.1 by using Bonk's criterion and prove Theorem 1.3 by combining Bonk's criterion and Mañé-Yin's characterization on semi-hyperbolic rational maps. In the last section, using the combinatorial method and renormalization theory, we construct a critically-infinite semi-hyperbolic rational map whose Julia set is quasisymmetrically equivalent to a round carpet.

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## 2 Some distortion estimations

In this section, we give some distortion estimations and useful lemmas, which will be used in the next section. We use  $\mathbb{D} := \{z : |z| < 1\}$  to denote the unit disk on the complex plane  $\mathbb{C}$ .

**Theorem 2.1** (Koebe's distortion theorem, [Pom, p. 9]). *Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be a univalent function. Then for every  $z \in \mathbb{D}$ , one has*

$$|f'(0)| \frac{|z|}{(1+|z|)^2} \leq |f(z) - f(0)| \leq |f'(0)| \frac{|z|}{(1-|z|)^2}; \quad \text{and} \quad (2.1)$$

$$|f'(0)| \frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq |f'(0)| \frac{1+|z|}{(1-|z|)^3}. \quad (2.2)$$

Let  $A$  be an annulus with non-degenerated boundary components. Then there exists a conformal map sending  $A$  to a standard annulus  $\{z \in \mathbb{C} : 0 < r < |z| < 1\}$ , where  $r > 0$  is uniquely determined by  $A$ . As an invariant under conformal maps, the *modulus* of  $A$  is defined as  $\text{mod}(A) = \frac{1}{2\pi} \log(1/r)$ . A set in  $\widehat{\mathbb{C}}$  is called a *Jordan disk* if it is homeomorphic to the unit disk  $\mathbb{D}$  and its boundary is a Jordan curve. Let  $A$  and  $B$  be two open sets in  $\widehat{\mathbb{C}}$ . We use the notation ' $A \Subset B$ ' if the closure  $\overline{A}$  is contained in  $B$ .

**Lemma 2.2.** *Let  $U_i \Subset V_i \neq \mathbb{C}$  be a pair of Jordan disks, where  $i = 1, 2$ . Suppose that  $\text{mod}(V_2 \setminus \overline{U}_2) \geq m > 0$  and  $f : V_1 \rightarrow V_2$  a conformal map with  $f(U_1) = U_2$ . Then there exists a constant  $C(m) \geq 1$  depending only on  $m$  such that for any  $x, y, z, w \in \overline{U}_1$ , one has*

$$\frac{1}{C(m)} \frac{|x-y|}{|z-w|} \leq \frac{|f(x)-f(y)|}{|f(z)-f(w)|} \leq C(m) \frac{|x-y|}{|z-w|}.$$

*Proof.* The proof is based on applying Koebe's distortion theorem. Without loss of generality, suppose that  $x \neq y$  and  $z \neq w$  are contained in the interior of  $U_1$ . If not, we can enlarge  $U_1$  appropriately. By Riemann's mapping theorem, there exists a conformal mapping  $g : (\Omega, \mathbb{D}) \rightarrow (U_1, V_1)$  which maps the unit disk  $\mathbb{D}$  onto  $V_1$  and a simply connected domain  $\Omega$  onto  $U_1$ . In particular, we require that  $g(0) = x$ .

We claim that there exists a positive constant  $r := r(m) < 1$  depending only on  $m$  such that  $\Omega \subset \mathbb{D}_r := \{z : |z| < r\}$ . Let  $\zeta \in \partial\Omega$  be the farthest point such that  $\text{dist}(0, \partial\Omega) = |\zeta|$ . Then  $\mathbb{D} \setminus \overline{\Omega}$  is an annulus separating 0 and  $\zeta$  from the unit circle. By Grötzsch's module theorem [LV, p. 54], we have

$$m \leq \text{mod}(V_1 \setminus \overline{U}_1) = \text{mod}(\mathbb{D} \setminus \overline{\Omega}) \leq \mu(|\zeta|),$$

where  $r \mapsto \mu(r)$  is a continuous and strictly decreasing function defined on the interval  $(0, 1)$ . This means that  $|\zeta| \leq \mu^{-1}(m)$  and the claim follows if we set  $r = \mu^{-1}(m)$ .

Now we consider  $f \circ g : \mathbb{D} \rightarrow V_2$  and  $g : \mathbb{D} \rightarrow V_1$ . For every  $\eta \in \Omega$ , by using (2.2) in Theorem 2.1, we have

$$|f'(x)| |g'(0)| \frac{1-r}{(1+r)^3} \leq |(f \circ g)'(\eta)| = |f'(g(\eta))| |g'(\eta)| \leq |f'(x)| |g'(0)| \frac{1+r}{(1-r)^3}. \quad (2.3)$$

Also, we have

$$|g'(0)| \frac{1-r}{(1+r)^3} \leq |g'(\eta)| \leq |g'(0)| \frac{1+r}{(1-r)^3}. \quad (2.4)$$

Combine (2.3) and (2.4), it follows that for every  $\xi \in U_1$ , we have

$$|f'(x)| \frac{(1-r)^4}{(1+r)^4} \leq |f'(\xi)| \leq |f'(x)| \frac{(1+r)^4}{(1-r)^4}. \quad (2.5)$$

Therefore, for  $x, y, z, w \in U_1$ , by (2.5), we have

$$|f(x) - f(y)| \leq \frac{(1+r)^4}{(1-r)^4} |f'(x)| \cdot |x - y| \text{ and } |f(z) - f(w)| \geq \frac{(1-r)^4}{(1+r)^4} |f'(x)| \cdot |z - w|.$$

Set  $C(m) = (1 + r(m))^8 / (1 - r(m))^8$ . The proof is complete.  $\square$

Let  $U$  be a hyperbolic disk in  $\mathbb{C}$  and  $E$  a connected and compact subset of  $U$  containing at least two points. For any  $z_1, z_2 \in E$ , the *turning* of  $E$  about  $z_1$  and  $z_2$  is defined by

$$\Lambda(E; z_1, z_2) = \frac{\text{diam}(E)}{|z_1 - z_2|}.$$

It is easy to see that  $1 \leq \Lambda(E; z_1, z_2) \leq \infty$  and  $\Lambda(E; z_1, z_2) = \infty$  if and only if  $z_1 = z_2$ .

By definition (see for example, [LV, p. 100]), a Jordan curve  $C$  is called a *quasicircle* if there exists a positive constant  $K \geq 1$  such that for any different points  $x, y \in C$ , the turning of  $\gamma$  about  $x$  and  $y$  satisfies

$$\Lambda(\gamma; x, y) \leq K,$$

where  $\gamma$  is one of the two components of  $C \setminus \{x, y\}$  with smaller diameter.

**Lemma 2.3.** *Let  $U_i \Subset V_i \neq \mathbb{C}$  be a pair of Jordan disks, where  $i = 1, 2$ . Suppose that  $\text{mod}(V_2 \setminus \overline{U}_2) \geq m > 0$  and  $f : V_1 \rightarrow V_2$  a conformal map with  $f(U_1) = U_2$ . If  $\partial U_2$  is a  $K$ -quasicircle, then there is a constant  $C(K, m) \geq 1$  such that  $\partial U_1$  is a  $C(K, m)$ -quasicircle.*

*Proof.* By definition, if  $\partial U_2$  is a  $K$ -quasicircle, then there exists a constant  $C(K) > 0$  such that for any different points  $z_1, z_2 \in \partial U_2$ , the turning of  $\gamma$  about  $z_1$  and  $z_2$  satisfies

$$\Lambda(\gamma; z_1, z_2) = \frac{\text{diam}(\gamma)}{|z_1 - z_2|} \leq C(K), \quad (2.6)$$

where  $\gamma$  is one of the component of  $\partial U_2 \setminus \{z_1, z_2\}$  with smaller diameter.

Let  $x, y \in \partial U_1$  be two different points which divide the quasicircle  $\partial U_1$  into two closed subcurves  $\alpha$  and  $\beta$ . Without loss of generality, let  $\alpha \subset \partial U_1$  be the subcurve with smaller diameter. Moreover, let  $z, w \in \alpha$  such that  $\text{diam}(\alpha) = |z - w|$ . By Lemma 2.2, we have

$$\Lambda(\alpha; x, y) = \frac{|z - w|}{|x - y|} \leq C(m) \frac{|f(z) - f(w)|}{|f(x) - f(y)|}, \quad (2.7)$$

where  $C(m)$  is the constant appeared in Lemma 2.2. Note that  $f(x), f(y)$  divide the quasicircle  $\partial U_2$  into two parts  $f(\alpha)$  and  $f(\beta)$ .

If  $\text{diam}(f(\alpha)) \leq \text{diam}(f(\beta))$ , then by (2.6) and (2.7), we have

$$\Lambda(\alpha; x, y) \leq C(m) \frac{\text{diam}(f(\alpha))}{|f(x) - f(y)|} \leq C(m)C(K). \quad (2.8)$$

If  $\text{diam}(f(\alpha)) > \text{diam}(f(\beta))$ , let  $z', w' \in \beta$  such that  $\text{diam}(\beta) = |z' - w'|$ . By (2.6) and Lemma 2.2, we have

$$\begin{aligned} \Lambda(\alpha; x, y) &\leq \Lambda(\beta; x, y) \leq \frac{|z' - w'|}{|x - y|} \leq C(m) \frac{|f(z') - f(w')|}{|f(x) - f(y)|} \\ &\leq C(m) \frac{\text{diam}(f(\beta))}{|f(x) - f(y)|} \leq C(m)C(K). \end{aligned} \quad (2.9)$$

Combine (2.8) and (2.9), the Lemma follows.  $\square$

Recall that the *relative distance*  $\Delta(A, B)$  of two subsets  $A$  and  $B$  in  $\widehat{\mathbb{C}}$  is defined in (1.1). Now we prove that relative distance of two disjoint Jordan curves can be controlled by the modulus of the annulus between them.

**Lemma 2.4** (Modulus controls the relative distance). *Let  $A \subset \widehat{\mathbb{C}}$  be an annulus with two boundary components  $C_1$  and  $C_2$ . If the modulus of  $A$  satisfies  $\text{mod}(A) \geq m > 0$ , then there exists a constant  $C(m) > 0$  depending only on  $m$  such that the relative distance of  $C_1$  and  $C_2$  satisfies  $\Delta(C_1, C_2) \geq C(m) > 0$ .*

*Proof.* Without loss of generality, we assume that  $A \subset \mathbb{C}$ ,  $C_1, C_2$  are not singletons and  $0 < \text{diam}(C_1) \leq \text{diam}(C_2)$  and

$$\text{dist}(C_1, C_2) = |x - y| \quad (2.10)$$

for  $x \in C_1$  and  $y \in C_2$ . There exists a point  $z \neq x$  in  $C_1$  such that  $|x - z| = \sup_{a \in C_1} |a - x|$ . Therefore, we have

$$\text{diam}(C_1) \leq 2|x - z|. \quad (2.11)$$

Consider the linear function  $h(t) = (t - x)/(x - z)$ , which maps  $x, y, z$  to  $0, (y - x)/(x - z)$  and  $-1$ . Then  $h(A)$  is an annulus separating the points  $0$  and  $-1$  from  $h(y)$  and  $\infty$ , respectively. Let

$$R = |h(y)| = |(y - x)/(x - z)|.$$

By Teichmüller's Module Theorem (see for example, [LV, p. 56]), we have

$$m \leq \text{mod}(A) = \text{mod}(h(A)) \leq 2\mu \left( \sqrt{\frac{1}{1+R}} \right),$$

where  $r \mapsto \mu(r)$  is a continuous and strictly decreasing map defined on the interval  $(0, 1)$ . By (2.10) and (2.11), this means that the relative distance of  $C_1$  and  $C_2$  is

$$\Delta(C_1, C_2) = \frac{\text{dist}(C_1, C_2)}{\text{diam}(C_1)} \geq \frac{|x - y|}{2|x - z|} = \frac{R}{2} \geq \frac{1}{2} \left( \frac{1}{(\mu^{-1}(m/2))^2} - 1 \right) := C(m).$$

The proof is complete.  $\square$

**Lemma 2.5** ([KL, Lemma 4.5]). *Let  $U_i \Subset V_i \neq \mathbb{C}$  be a pair of Jordan disks, where  $i = 1, 2$ . Suppose that  $g : V_1 \rightarrow V_2$  is a proper holomorphic map of degree  $d \geq 1$  and  $U_1$  is a component of  $g^{-1}(U_2)$ . Then*

$$\text{mod}(V_1 \setminus \overline{U}_1) \leq \text{mod}(V_2 \setminus \overline{U}_2) \leq d \text{mod}(V_1 \setminus \overline{U}_1).$$

Let  $U$  be a hyperbolic disk in  $\mathbb{C}$  and  $z \in U$ . The *shape* of  $U$  about  $z$ , denoted by  $\text{Shape}(U, z)$ , is defined as

$$\text{Shape}(U, z) = \frac{\max_{w \in \partial U} |w - z|}{\min_{w \in \partial U} |w - z|} = \frac{\max_{w \in \partial U} |w - z|}{\text{dist}(z, \partial U)}.$$

It is obvious that  $\text{Shape}(U, z) = \infty$  if and only if  $U$  is unbounded and  $\text{Shape}(U, z) = 1$  if and only if  $U$  is a round disk centered at  $z$ . In other cases,  $1 < \text{Shape}(U, z) < \infty$ .

**Lemma 2.6** ([QWY, Lemma 6.1]). *Let  $U_i \Subset V_i \neq \mathbb{C}$  be a pair of Jordan disks with  $\text{mod}(V_2 \setminus \overline{U}_2) \geq m > 0$ , where  $i = 1, 2$ . Suppose that  $g : V_1 \rightarrow V_2$  is a proper holomorphic map of degree  $d \geq 1$  and  $U_1$  is a component of  $g^{-1}(U_2)$ . Then there are two positive constants  $C_1(d, m)$  and  $C_2(d, m)$  depending only on  $d$  and  $m$ , such that*

(1) *For all  $z \in U_1$ , the shape satisfies*

$$\text{Shape}(U_1, z) \leq C_1(d, m) \text{Shape}(U_2, g(z)).$$

(2) For any connected and compact subset  $E$  of  $U_1$  with the cardinal number  $\sharp E \geq 2$  and any  $z_1, z_2 \in E$ , the turning satisfies

$$\Lambda(E; z_1, z_2) \leq C_2(d, m) \Lambda(g(E); g(z_1), g(z_2)).$$

Lemma 2.6 means that the shape and the turning of the interior boundary of an annulus can be controlled under a proper holomorphic map if the modulus of this annulus has a lower bound.

### 3 Proofs of the Main Theorems

If a rational map  $f$  whose Julia set  $J_f$  is a Sierpiński carpet, then  $f$  cannot be a polynomial. In fact, the intersection of the closure of the bounded Fatou components (if any) and the basin of infinity of  $f$  is non-empty provided  $f$  is a polynomial since the Julia set  $J_f$  is the boundary of the basin of infinity. If we want to prove Theorem 1.1, we need to prove that the peripheral circles of the carpets are uniform quasicircles and uniformly relatively separated by Bonk's criterion.

**3.1. Mañé's Theorem and a lemma.** We first give a theorem due to Mañé, which will be used frequently later.

**Theorem 3.1** ([Ma, Theorem II]). *Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a rational map with degree at least two. If a point  $x \in J_f$  is not a parabolic periodic point and is not contained in the  $\omega$ -limit set of a recurrent critical point, then for any  $\epsilon > 0$  there exists an open neighborhood  $U_x$  of  $x$  such that:*

- (C1) *For all  $n \geq 0$ , every component of  $f^{-n}(U_x)$  has diameter  $\leq \epsilon$ ;*
- (C2) *There exists  $d > 0$  such that for all  $n \geq 0$  and every connected component  $V$  of  $f^{-n}(U_x)$ , the degree of  $f^n : V \rightarrow U_x$  is  $\leq d$ .*

When we pull back a Jordan disk  $U$  by a rational map  $f$ , there maybe exist a component  $W$  of  $f^{-1}(U)$  which is not simply connected. If the boundary  $\partial U$  avoids the critical values, then  $\partial W$  is the union of finitely many disjoint Jordan curves  $\{C_i\}$ . Moreover, we have  $f(C_i) = \partial U$  for each  $i$ . Note that  $W$  is a connected set whose boundary consists of finitely many Jordan curves. We have  $\widehat{\mathbb{C}} \setminus \overline{W} = \bigcup_i V_i$ , where each  $V_i$  is a Jordan disk bounded by the Jordan curve  $C_i$ . Since the restriction of  $f$  on  $V_i$  is a holomorphic branched covering and  $f(\partial V_i) = \partial U$ , we have  $f(V_i) = \widehat{\mathbb{C}}$  or  $f(V_i) = \widehat{\mathbb{C}} \setminus \overline{U}$ . In other words, the image of each component of the complement of  $W$  under  $f$  is either  $\widehat{\mathbb{C}}$  or  $\widehat{\mathbb{C}} \setminus \overline{U}$ . See Figure 2 for an example.

In the rest of this paper, we only consider the rational maps whose Julia sets are not the whole complex sphere. Therefore, after conjugating  $f$  by a suitable Möbius transformation, we always assume that  $\infty$  lies in the Fatou set. This means that  $J_f$  is a compact set in  $\mathbb{C}$ . In the following, we equip  $J_f$  the Euclidean metric if not special specified. We use  $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$  to denote the round disk in  $\mathbb{C}$  with the center  $a \in \mathbb{C}$  and radius  $r > 0$ .

**Lemma 3.2.** *Let  $f$  be a rational map with degree at least two and  $J_f \subset \mathbb{C}$ . Suppose that  $x \in J_f$  is not a parabolic periodic point and is not contained in the  $\omega$ -limit set of a recurrent critical point. Then there exists an open neighborhood  $U_x$  of  $x$  such that*

- (C3) *For all  $n \geq 0$ , every connected component of  $f^{-n}(U_x)$  is simply connected.*

*Proof.* By the assumption that  $\infty \notin J_f$ , the grand orbit of  $\infty$  lies in the Fatou set of  $f$ . Let  $\delta_0 > 0$  be a small positive number such that

$$0 < \delta_0 \leq \text{dist}(f^{-1}(\infty), J_f)/2. \quad (3.1)$$



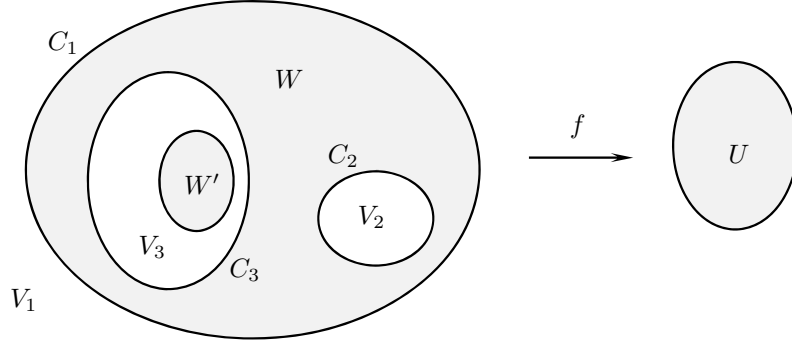


Figure 2: The pull back of a simply connected domain  $U$  under the rational map  $f$  with degree 4, where  $f(W) = U$  and  $\partial W = C_1 \cup C_2 \cup C_3$ . The complement of  $\overline{W}$  consists of 3 simply connected components  $V_1$ ,  $V_2$  and  $V_3$ . In particular,  $f(V_1) = f(V_2) = \widehat{\mathbb{C}} \setminus \overline{U}$  and  $f(V_3) = \widehat{\mathbb{C}}$ . Moreover,  $W$  contains 4 critical points of  $f$  and  $V_3 \setminus W'$  (the white annulus) contains two.

By Theorem 3.1, there exists an open neighborhood  $U'_x$  of  $x$  such that every component of  $f^{-n}(U'_x)$  has diameter  $\leq \delta_0$  for all  $n \geq 0$ .

Let  $U_x := \mathbb{D}(x, \delta_x)$  be the largest round disk which is contained in  $U'_x$ . We claim that every component  $W_n$  of  $f^{-n}(U_x)$  is simply connected. If not, let  $V_n$  be a bounded component of  $\mathbb{C} \setminus \overline{W}_n$ , where  $n \geq 1$ . Then  $\partial V_n \subset \partial W_n$  and so  $\text{diam}(V_n) \leq \delta_0$ . This means that  $V_n$  cannot intersect  $f^{-1}(\infty)$ . Inductively, one can easily check that  $f^{\circ k}(V_n) \cap f^{-1}(\infty) = \emptyset$  for  $0 \leq k \leq n-1$ . It follows that  $\infty \notin f^{\circ n}(V_n)$ , which is a contradiction since  $f^{\circ n}(V_n) = \widehat{\mathbb{C}}$  or  $f^{\circ n}(V_n) = \widehat{\mathbb{C}} \setminus \overline{U}_x$ . Therefore, such  $V_n$  does not exist. This means that  $W_n$  is simply connected. The proof is complete.  $\square$

Lemma 3.2 is useful in the following since we need to obtain the simply connected preimages of a simply connected domain.

**3.2. Sufficiency for the property of uniform quasicircles.** In this subsection, we prepare some lemmas and give two sufficient conditions such that the boundaries of the Fatou components are uniform quasicircles. We first discuss the regularity of the boundaries of the periodic Fatou components and then spread the results to their all preimages.

**Lemma 3.3.** *Let  $\Gamma$  be a Jordan curve in the plane  $\mathbb{C}$ . Then there exists a constant  $\delta_\Gamma > 0$  depending only on  $\Gamma$  such that, for any Jordan subarc  $\gamma \subset \Gamma$  with  $\text{diam}(\gamma) \leq \delta_\Gamma$ , one has  $\text{diam}(\gamma) < \text{diam}(\Gamma \setminus \gamma)$ .*

*Proof.* Consider the function  $h : \Gamma \times \Gamma \rightarrow \mathbb{R}$  which is defined by  $h(x, y) = \text{diam}(L'(x, y))$ , where  $L'(x, y)$  is one of the two components of  $\Gamma \setminus \{x, y\}$  with larger diameter. Obviously, the map  $h$  is continuous. Since  $\Gamma \times \Gamma$  is compact, the function  $h$  has a minimum  $\delta' > 0$ . Then the lemma holds if we set  $\delta_\Gamma = \delta'/2$ .  $\square$

**Lemma 3.4.** *Let  $f$  be a rational map with degree at least two and  $U$  a Fatou component which is a Jordan disk. Then  $f|_{\partial U}$  is a local homeomorphism.*

*Proof.* The image  $V = f(U)$  is a Fatou component and hence a domain. Since  $f$  maps the boundary of  $U$  to that of  $V$ , it follows that  $V$  is a Jordan disk as well and  $f(\partial U) = \partial V$ . For an annulus  $A$  with the outer boundary  $\partial V$  and the inner boundary surrounding all the critical values in  $V$ , then  $A' = (f|_U)^{-1}(A)$  is also an annulus in  $U$  with the outer boundary coinciding

with  $\partial U$  by Riemann-Hurwitz's formula. Then  $f : A' \rightarrow A$  is an unbranched covering. Thus the restriction of  $f$  on  $\partial U$  is a local homeomorphism.  $\square$

**Lemma 3.5** (The boundaries of periodic Fatou components are quasicircles). *Let  $f$  be a rational map with degree at least two. Suppose that  $U$  is a periodic Fatou component of  $f$  whose boundary  $\partial U$  is a Jordan curve and  $\partial U$  contains neither parabolic periodic points nor the points in  $\omega(c)$  for any recurrent critical point  $c$ . Then  $\partial U$  is a quasicircle.*

*Proof.* After iterating  $f$  by several times, we can assume that the periodic Fatou component  $U$  is fixed by  $f$ . Without loss of generality, we suppose that  $\infty \notin J_f$ . Let  $\epsilon = \delta_0 > 0$  be the number defined as in (3.1). For any  $x \in \partial U$ , by Theorem 3.1 and Lemma 3.2, there exists an open neighborhood  $U_x := \mathbb{D}(x, \delta_x)$  of  $x$  satisfying (C1), (C2) and (C3). Since  $\partial U$  is compact and  $\partial U \subseteq \bigcup_{x \in \partial U} \mathbb{D}(x, \delta_x/2)$ , one can select a collection of finite number of elements  $\mathcal{U} = \{\mathbb{D}(x_1, \delta_{x_1}/2), \dots, \mathbb{D}(x_N, \delta_{x_N}/2)\}$  such that  $\partial U$  is covered by  $\mathcal{U}$ . Let  $\delta_1 > 0$  be the Lebesgue number of  $\mathcal{U}$ . Then every subset of  $\partial U$  with diameter  $\leq \delta_1$  must be contained in at least one open disk  $\mathbb{D}(x_i, \delta_{x_i}/2)$  for some  $1 \leq i \leq N$ .

By Lemma 3.4, the restriction of  $f$  on  $\partial U$  is a local homeomorphism. This means that there exists a number  $\delta_2 > 0$  such that for any subset  $E \subset \partial U$  with  $\text{diam}(E) \leq \delta_2$ , the restriction of  $f$  on  $E$  is a homeomorphism. Recall that  $\delta_{\partial U} > 0$  is the number depending only on  $\partial U$  which is defined in Lemma 3.3. We define

$$\delta := \min \left\{ \frac{\delta_1}{M}, \delta_2, \frac{\delta_{\partial U}}{M} \right\}, \quad (3.2)$$

where  $M := 1 + \sup\{|f'(z)| : \text{dist}(z, J_f) \leq \delta_0\} < +\infty$ .

Let  $x, y$  be two different points in  $\partial U$ . We use  $\gamma := L(x, y)$  to denote one of the two components of  $\partial U \setminus \{x, y\}$  with the smaller diameter. Now we divide the argument into two cases.

**Case 1:** Suppose that  $\text{diam}(\gamma) \geq \delta$ . Define  $E := \{(\xi, \eta) \in \partial U \times \partial U : \text{diam}(L(\xi, \eta)) \geq \delta\}$ . Then  $E$  is compact and  $(\xi, \xi) \notin E$ . The function

$$h : \partial U \times \partial U \rightarrow \mathbb{R}^+ \text{ defined by } (\xi, \eta) \mapsto \frac{\text{diam}(L(\xi, \eta))}{|\xi - \eta|}$$

is continuous on  $E$ . Then  $h$  has a maximum  $K_1$  on  $E$  since  $E$  is compact. In particular, the turning of  $\gamma$  about  $x$  and  $y$  satisfies

$$\Lambda(\gamma; x, y) = \frac{\text{diam}(\gamma)}{|x - y|} \leq K_1. \quad (3.3)$$

**Case 2:** Suppose that  $\text{diam}(\gamma) < \delta$ . Denote  $\gamma_n := f^{\circ n}(\gamma)$  for  $n \geq 0$ . Note that the forward orbit of  $\gamma$  will eventually cover  $\partial U$ . There is a smallest integer  $n \geq 0$  such that

$$\text{diam}(\gamma_n) < \delta \text{ and } \text{diam}(\gamma_{n+1}) = \text{diam}(f(\gamma_n)) \geq \delta. \quad (3.4)$$

By the choice of  $\delta$  in (3.2), we know that  $f^{\circ(n+1)}|_{\gamma}$  is a homeomorphism and so  $\gamma_{n+1}$  is a Jordan arc connecting  $f^{\circ(n+1)}(x)$  and  $f^{\circ(n+1)}(y)$ . Note that there exist two points  $z_1, z_2 \in \gamma_n$ , such that

$$\begin{aligned} \text{diam}(\gamma_{n+1}) &= |f(z_1) - f(z_2)| \leq \int_{[z_1, z_2]} |f'(z)| |dz| \\ &\leq M|z_1 - z_2| \leq M \text{diam}(\gamma_n) \leq M\delta \leq \min\{\delta_1, \delta_{\partial U}\}, \end{aligned} \quad (3.5)$$

where  $[z_1, z_2]$  is the straight segment connecting  $z_1$  and  $z_2$ .

By the definition of  $\delta_{\partial U}$  and Lemma 3.3, the Jordan arc  $\gamma_{n+1}$  is one of the two components of  $\partial U \setminus \{f^{\circ(n+1)}(x), f^{\circ(n+1)}(y)\}$  with smaller diameter. Since  $\text{diam}(\gamma_{n+1}) \geq \delta$  by (3.4), as discussed in Case 1 above, we have

$$\Lambda(\gamma_{n+1}; f^{\circ(n+1)}(x), f^{\circ(n+1)}(y)) \leq K_1. \quad (3.6)$$

By the definition of  $\delta_1$ , there exists a disk  $\mathbb{D}(x_i, \delta_{x_i}/2)$  such that  $\gamma_{n+1} \subset \mathbb{D}(x_i, \delta_{x_i}/2)$  for some  $1 \leq i \leq N$  since  $\text{diam}(\gamma_{n+1}) \leq \delta_1$  by (3.5). Let  $B_{n+1}(x_i, \delta_{x_i}/2)$  and  $B_{n+1}(x_i, \delta_{x_i})$ , respectively, be the components of  $f^{-n-1}(\mathbb{D}(x_i, \delta_{x_i}/2))$  and  $f^{-n-1}(\mathbb{D}(x_i, \delta_{x_i}))$  both containing  $\gamma$ . Note that both of them are simply connected by the choice of  $\delta_{x_i}$ . Applying Lemma 2.6 to the case  $(U_1, V_1) = (B_{n+1}(x_i, \delta_{x_i}/2), B_{n+1}(x_i, \delta_{x_i}))$ ,  $(U_2, V_2) = (\mathbb{D}(x_i, \delta_{x_i}/2), \mathbb{D}(x_i, \delta_{x_i}))$  and  $m = \frac{1}{2\pi} \log 2$ , together with (3.6), we have

$$\Lambda(\gamma; x, y) \leq C_2(d_i) \Lambda(\gamma_{n+1}; f^{\circ(n+1)}(x), f^{\circ(n+1)}(y)) \leq C_2(d_i) K_1, \quad (3.7)$$

where  $C_2(d_i)$  is a constant depending only on  $d_i$  and  $d_i > 0$  is the number appeared in Theorem 3.1 which depends on  $x_i$ . Let

$$K = K_1(1 + \max_{1 \leq i \leq N} C_2(d_i)).$$

Then  $\Lambda(\gamma; x, y) \leq K$  holds for any different  $x, y \in \partial U$  by (3.3) and (3.7). By the arbitrariness of  $x$  and  $y$ , this means that  $\partial U$  is a quasicircle. The proof is completed.  $\square$

Now we need to consider when the boundaries of all the Fatou components are uniform quasicircles. According to Sullivan [Sul], each Fatou component of a rational map is eventually periodic. It is natural to consider the pull back of the periodic Fatou components and then using some distortion lemmas to control the shape of pre-periodic Fatou components. To do this, it is necessary to construct a larger simply connected domain surrounding the periodic Fatou component such that all components of its preimages under the  $n$ -th iteration are still simply connected.

**Proposition 3.6** (Uniform quasicircles I). *Let  $f$  be a rational map such that the boundary of each Fatou component is a Jordan curve. Suppose that all the boundaries of periodic Fatou components are disjoint with the  $\omega$ -limit sets of the critical points. Then the boundaries of all the Fatou components of  $f$  are uniform quasicircles.*

*Proof.* If all periodic Fatou components of  $f$  are disjoint with the  $\omega$ -limit sets of the critical points, then  $f$  has no parabolic periodic points. By Lemma 3.5 and Sullivan's eventually periodic theorem, all the boundaries of the Fatou components of  $f$  are quasicircles. We only need to prove that they are uniform quasicircles.

Let  $\mathcal{U}'$  be the collection of all the Fatou components such that each of them is either a critical Fatou component (contains at least one critical point) or a periodic Fatou component. We use  $\mathcal{U} := O^+(\mathcal{U}') = \{U_1, \dots, U_n\}$  to denote the union of the forward orbits of all the Fatou components in  $\mathcal{U}'$ . Note that the number of Fatou components in  $\mathcal{U}$  is finite since  $\mathcal{U}'$  is. Therefore, there exists a constant  $K_1 > 1$  such that each  $\partial U_i$  is a  $K_1$ -quasicircle. For each  $1 \leq i \leq n$ , let  $V_i$  be a Jordan disk such that  $V_i \setminus \overline{U_i}$  is an annulus which is disjoint with the  $\omega$ -limit sets of the critical points.

Let  $\text{mod}(V_i \setminus \overline{U_i}) = m_i > 0$  for  $1 \leq i \leq n$ . For each Fatou component  $U \notin \mathcal{U}$ , there exists a minimal number  $k \geq 1$  such that  $f^{\circ k}(U) = U_i \in \mathcal{U}$  for some  $i$ . Let  $V$  be the component of  $f^{-k}(V_i)$  containing  $U$ . Then  $f^{\circ k} : V \rightarrow V_i$  is conformal and  $V$  is a Jordan disk since  $V_i$  contains no points in the critical orbits. By Lemma 2.3, the boundary  $\partial U$  is a  $C(K_1, m_i)$ -quasicircle, where  $C(K_1, m_i)$  is a constant depending only on  $K_1$  and  $m_i$ .

Let  $K = \max_{1 \leq i \leq n} C(K_1, m_i)$ . Then the boundary of each Fatou component of  $f$  is a  $K$ -quasircle. By the arbitrariness of  $U$ , this means that the Fatou components of  $f$  are uniform quasircles. The proof is completed.  $\square$

Recall that a rational map  $f$  is called *semi-hyperbolic* if and only if the Julia set  $J_f$  contains neither parabolic periodic points nor recurrent critical points.

**Proposition 3.7** (Uniform quasircles II). *Let  $f$  be a semi-hyperbolic rational map such that the boundary of each Fatou component is a Jordan curve. Then the boundaries of all the Fatou components of  $f$  are uniform quasircles.*

*Proof.* By Lemma 3.5 and Sullivan's eventually periodic theorem, it follows that all the boundaries of the Fatou components of  $f$  are quasircles since  $f$  is semi-hyperbolic. We only need to prove that they are uniform quasircles. According to [Yin, Theorem 1.2], the Julia set  $J_f$  is locally connected. Then for any  $\epsilon > 0$ , there are only finitely many Fatou components with diameter  $\geq \epsilon$  [Mi3, Lemma 19.5].

Without loss of generality, we suppose that  $\infty \notin J_f$ . Let  $\epsilon = \delta_0 > 0$  be the number defined as in (3.1). For any  $x \in J_f$ , by Theorem 3.1 and Lemma 3.2, there exists an open neighborhood  $U_x := \mathbb{D}(x, \delta_x)$  of  $x$  satisfying (C1), (C2) and (C3). Since  $J_f$  is compact, there exists a collection of finite number of elements  $\mathcal{U} = \{\mathbb{D}(x_1, \delta_{x_1}/2), \dots, \mathbb{D}(x_N, \delta_{x_N}/2)\}$  such that  $J_f$  is covered by  $\mathcal{U}$ . We use  $\delta > 0$  to denote the Lebesgue number of  $\mathcal{U}$ . Then every subset of  $J_f$  with diameter  $\leq \delta$  must be contained in at least one open disk  $\mathbb{D}(x_i, \delta_{x_i}/2)$  for some  $1 \leq i \leq N$ .

We divide the collection of all the Fatou components  $\mathcal{F}$  of  $f$  into two classes as following. Let  $\mathcal{F}_0$  be the collection of all the Fatou components such that each  $U \in \mathcal{F}_0$  is one of the following cases: (1)  $U$  contains at least one critical point; (2)  $U$  is periodic; (3)  $\text{diam}(U) \geq \delta$ . Let  $\mathcal{F}'_1 := O^+(\mathcal{F}_0)$  be the set of the union of the forward orbits of all the Fatou components in  $\mathcal{F}_0$ . Define  $\mathcal{F}_1 := \mathcal{F}'_1 \cup f^{-1}(\mathcal{F}'_1)$ . By Sullivan's eventually periodic theorem, the number of Fatou components in  $\mathcal{F}_1$  is finite since  $\mathcal{F}_0$  is also. Therefore, there exists a constant  $K_1 > 1$  such that each Fatou component in  $\mathcal{F}_1$  is a  $K_1$ -quasircle.

For any Fatou component  $U \in \mathcal{F} \setminus \mathcal{F}_1$ , we have  $\text{diam}(U) < \delta$ . There exists a minimal  $n_U \geq 1$  such that  $f^{n_U}(U) \in f^{-1}(\mathcal{F}'_1) \setminus \mathcal{F}'_1 \subset \mathcal{F}_1$  and  $\text{diam}(f^{n_U}(U)) < \delta$ . Moreover, the map  $f^{n_U} : U \rightarrow f^{n_U}(U)$  is conformal. By the definition of  $\delta$ , there exists some disk  $\mathbb{D}(x_i, \delta_{x_i}/2)$  in  $\mathcal{U}$  such that  $f^{n_U}(U) \subset \mathbb{D}(x_i, \delta_{x_i}/2)$ . We use  $B_U$  and  $B'_U$ , respectively, to denote the components of  $f^{-n_U}(\mathbb{D}(x_i, \delta_{x_i}/2))$  and  $f^{-n_U}(\mathbb{D}(x_i, \delta_{x_i}))$  both containing  $U$ .

Let  $x, y \in \partial U$  be two different points such that  $\partial U \setminus \{x, y\} = \gamma_1 \cup \gamma_2$ . Then  $f^{n_U}(\gamma_1)$  and  $f^{n_U}(\gamma_2)$  are both Jordan arcs connecting  $f^{n_U}(x)$  with  $f^{n_U}(y)$ . Applying Lemma 2.6 (2) to the case  $(U_1, V_1) = (B_U, B'_U)$ ,  $(U_2, V_2) = (\mathbb{D}(x_i, \delta_{x_i}/2), \mathbb{D}(x_i, \delta_{x_i}))$ ,  $m = \frac{1}{2\pi} \log 2$ ,  $g = f^{n_U}$  and  $E = \gamma_j$ , where  $j = 1, 2$ , we have

$$\Lambda(\gamma_j; x, y) \leq C_2(d_i) \Lambda(f^{n_U}(\gamma_j); f^{n_U}(x), f^{n_U}(y)),$$

where  $C_2(d_i)$  is a constant depending only on  $d_i$  and  $d_i > 0$  is the number appeared in Theorem 3.1 which depends on  $x_i$ . Then

$$\min_{j \in \{1, 2\}} \{\Lambda(\gamma_j; x, y)\} \leq C_2(d_i) \min_{j \in \{1, 2\}} \{\Lambda(f^{n_U}(\gamma_j); f^{n_U}(x), f^{n_U}(y))\} \leq C_2(d_i) K_1.$$

Let  $K = \max_{1 \leq i \leq N} C_2(d_i) K_1$ . Then  $\partial U$  is a  $K$ -quasircle by the arbitrariness of  $x$  and  $y$ . By the arbitrariness of  $U$ , we know that each Fatou component of  $f$  is a  $K$ -quasircle and  $K$  is a constant depending only on  $f$ . The proof is completed.  $\square$

Figure 1 shows a rational map having a parabolic periodic point whose Julia set is a Sierpiński carpet but the peripheral circles of  $J_f$  are not uniform quasicircles. Note that in Propositions 3.6 and 3.7, we do not require the closure of Fatou components are disjoint to each other. They can touch each other at the points on their boundaries. It seems that the conditions in Proposition 3.6 is much stronger than in Proposition 3.7. However, it is not true. One can construct a rational map with recurrent critical points, whose  $w$ -limit sets are disjoint with boundaries of Fatou components, using similar method as stated in Section 4.

**3.3. Sufficiency for the property of uniformly relatively separated.** By Lemma 2.4, if the lower bound of the annuli between the boundaries of the Fatou components can be controlled, then one can prove that the peripheral circles of the carpet Julia set are uniformly relatively separated.

**Proposition 3.8** (Uniformly relatively separated). *Let  $f$  be a rational map whose Julia set  $J_f$  is a Sierpiński carpet. If the boundaries of all periodic Fatou components contain no points in  $\omega(c)$  for any critical point  $c \in J_f$ , then the boundaries of Fatou components are uniformly relatively separated.*

*Proof.* After iterating  $f$  by some times, we can assume that all the periodic Fatou components  $X_1, \dots, X_n$  have period precise one. For  $1 \leq i \leq n$ , let  $Y_i$  be a simply connected domain containing  $X_i$  such that  $Y_1, \dots, Y_n$  are mutually disjoint and each annulus  $H_i := Y_i \setminus \overline{X_i}$  contains no points in the critical orbits. Define  $m = \min_{1 \leq i \leq n} \text{mod}(H_i) > 0$ .

For any two different Fatou components  $U_1$  and  $U_2$ , there exist two minimal numbers  $n_1, n_2 \geq 0$  such that  $f^{n_1}(U_1) = X_{k_1}$  and  $f^{n_2}(U_2) = X_{k_2}$  for  $1 \leq k_1, k_2 \leq n$ , where  $X_{k_1}, X_{k_2}$  are Fatou components with period one. Since  $H_{k_i}$  contains no critical values of  $f^{n_i}$  for  $i \in \{1, 2\}$  and so the restriction of  $f^{n_i}$  on each components of  $f^{-n_i}(H_{k_i})$  is an unbranched covering. By Riemann-Hurwitz's formula, it follows that each component of their preimages is an annulus. Therefore, there exist two simply connected domains  $V_1, V_2$  surrounding  $U_1, U_2$  such that  $V_i \setminus \overline{U_i}$  is a component of  $f^{-n_i}(H_{k_i})$  and  $\deg(f^{n_i} : V_i \rightarrow Y_{k_i}) = \deg(f^{n_i} : U_i \rightarrow X_{k_i})$ . Note that  $f^{o_{j_1}}(U_i) \cap f^{o_{j_2}}(U_i) = \emptyset$  for  $0 \leq j_1 < j_2 \leq n_i$  and  $f$  has only finitely many critical points. So the degree of  $f^{o_{j_1}}|_{U_i}$  is bounded by some number  $N \geq 1$  depending only on  $f$ . Denote by  $A$  the annulus bounded by  $\partial U_1$  and  $\partial U_2$  in  $\widehat{\mathbb{C}}$ . We now divide the arguments into two cases.

**Case 1:** Suppose that  $n_1 = n_2$ . Then  $V_1$  and  $V_2$  are two disjoint components of  $f^{-n_1}(Y_{n_1} \cup Y_{n_2})$ . By Lemma 2.5, we have

$$\text{mod}(A) \geq \text{mod}(V_1 \setminus \overline{U_1}) + \text{mod}(V_2 \setminus \overline{U_2}) \geq \text{mod}(H_{k_1})/N + \text{mod}(H_{k_2})/N \geq 2m/N.$$

**Case 2:** Suppose that  $n_1 > n_2$ . We claim that  $V_1$  and  $U_2$  are disjoint. Otherwise, the annulus  $V_1 \setminus \overline{U_1}$  intersects  $U_2$  and so  $f^{n_2}(V_1 \setminus \overline{U_1})$  intersects the fixed Fatou component  $X_{k_2}$ . Then  $H_{k_1} = f^{o(n_1-n_2)}(f^{n_2}(V_1 \setminus \overline{U_1}))$  joints with  $X_{k_2}$ , which contradicts with the choice of  $H_{k_1}$ . Then we have

$$\text{mod}(A) \geq \text{mod}(V_1 \setminus \overline{U_1}) \geq m/N.$$

Above all, the annulus  $A$  has modulus not less than  $m/N$ . By Lemma 2.4,  $U_1$  and  $U_2$  are relatively separated with the relative distance  $\Delta(\partial U_1, \partial U_2)$  depending only on  $m$  and  $N$ . By the arbitrariness of  $U_1$  and  $U_2$ , the peripheral circles of the carpet Julia set are uniformly relatively separated. The proof is completed.  $\square$

Note that the condition in Proposition 3.8 does not exclude the existence of parabolic points on the Julia set. Actually, the peripheral circles of the parabolic rational map appeared in Figure 1 are uniformly relatively separated.

**3.4. The property of non-uniformly relatively separated.** If the peripheral circles of a carpet Julia set are uniformly relatively separated, a natural question is whether it implies that all the boundaries of pre-periodic Fatou components avoid the accumulation points of the critical orbits in the Julia set. We give the answer in the following proposition.

**Proposition 3.9** (Non-uniformly relatively separated). *Let  $f$  be a semi-hyperbolic rational map whose Julia set is a Sierpiński carpet. Suppose that there exists a Fatou component  $U$  of  $f$  such that  $\partial U \cap \omega(c) \neq \emptyset$  for some critical point  $c \in J_f$ . Then the boundaries of Fatou components of  $f$  are not uniformly relatively separated.*

*Proof.* Without loss of generality, we suppose that  $\infty \notin J_f$ . Let  $\epsilon = \delta_0 > 0$  be the number defined as in (3.1). Let  $x \in \partial U \cap \omega(c)$ . By Theorem 3.1 and Lemma 3.2, there exists a number  $\delta_x > 0$  such that the open neighborhood  $U_x := \mathbb{D}(x, \delta_x)$  satisfies (C1), (C2) and (C3). Since  $\partial U \cap \omega(c) \neq \emptyset$  and  $J_f$  is a Sierpiński carpet, it follows that the forward orbit of  $c$  is infinite. Let  $c_{k_n} := f^{\circ k_n}(c)$  be the point in the forward orbit of  $c$  converging to  $x$ . Set  $\epsilon_{k_n} := |x - c_{k_n}|$ . We have  $\epsilon_{k_n} \rightarrow 0$  as  $n \rightarrow \infty$ . Given  $0 < \delta < \delta_x$ , there exists sufficiently large  $N$  such that  $\mathbb{D}(c_{k_n}, \delta/2) \subseteq \mathbb{D}(c_{k_n}, \delta) \subseteq \mathbb{D}(x, \delta_x)$  for any  $n \geq N$ .

Evidently, the round disks  $\mathbb{D}(c_{k_n}, \epsilon_{k_n})$ ,  $\mathbb{D}(c_{k_n}, \delta/2)$  and  $\mathbb{D}(c_{k_n}, \delta)$  satisfy (C1), (C2) and (C3) in Theorem 3.1 and Lemma 3.2. Pulling these three disks back by  $f^{\circ(k_n-1)}$  and  $f^{\circ k_n}$  respectively, we denote by  $X_{k_n-1}, Y_{k_n-1}, Z_{k_n-1}$ , respectively,  $X_{k_n}, Y_{k_n}, Z_{k_n}$  the simply connected components of their preimages containing the critical value  $c_1$  and the critical point  $c$  respectively. Let  $U_{k_n-1}$  be a component of  $f^{-(k_n-1)}(U)$  such that  $\partial X_{k_n-1} \cap \partial U_{k_n-1} \neq \emptyset$ . Then we can choose a point  $x_{k_n-1} \in \partial X_{k_n-1} \cap \partial U_{k_n-1}$ . Note that such  $x_{k_n-1}$  may be not unique. See Figure 3.

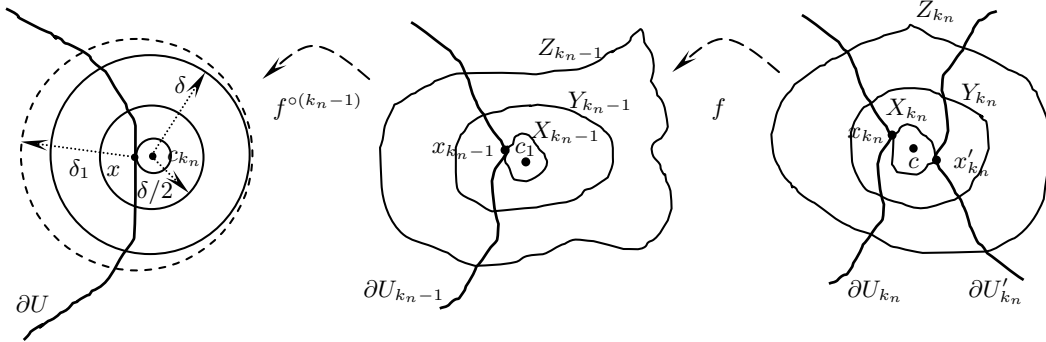


Figure 3: Sketch illustration of the mapping relation.

Since  $c_1$  is a critical value, there exist at least two different Fatou components  $U_{k_n}$  and  $U'_{k_n}$ , which are both the preimages of  $U_{k_n-1}$  such that  $\overline{X}_{k_n} \cap \overline{U}_{k_n} \neq \emptyset$  and  $\overline{X}_{k_n} \cap \overline{U}'_{k_n} \neq \emptyset$ . Let  $x_{k_n} \in \partial X_{k_n} \cap \partial U_{k_n}$  and  $x'_{k_n} \in \partial X_{k_n} \cap \partial U'_{k_n}$  be the preimages of  $x_{k_n-1}$ . We will show the relative distance between  $U_{k_n}$  and  $U'_{k_n}$  converges to zero as  $\epsilon_{k_n}$  converges to zero.

Applying the Lemma 2.6 (1) to the case  $(U_1, V_1) = (Y_{k_n}, Z_{k_n})$ ,  $(U_2, V_2) = (\mathbb{D}(c_{k_n}, \delta/2), \mathbb{D}(c_{k_n}, \delta))$  and  $g = f^{\circ k_n}$ , we know that there exists a constant  $C_1(d) > 0$  such that

$$\text{Shape}(Y_{k_n}, c) = \frac{\max_{w \in \partial Y_{k_n}} |w - c|}{\text{dist}(c, \partial Y_{k_n})} \leq C_1(d), \quad (3.8)$$

where  $d$  is the constant appeared in Theorem 3.1.

Similarly, there exists a constant  $C_2(d) > 0$  such that

$$\text{Shape}(X_{k_n}, c) = \frac{\max_{w \in \partial X_{k_n}} |w - c|}{\text{dist}(c, \partial X_{k_n})} \leq C_2(d). \quad (3.9)$$

Now we estimate the relative distance of  $\partial U_{k_n}$  and  $\partial U'_{k_n}$  by (3.8) and (3.9).

$$\begin{aligned}
 \Delta(\partial U_{k_n}, \partial U'_{k_n}) &= \frac{\text{dist}(\partial U_{k_n}, \partial U'_{k_n})}{\min\{\text{diam}(\partial U_{k_n}), \text{diam}(\partial U'_{k_n})\}} \\
 &\leq \frac{|x_{k_n} - x'_{k_n}|}{\text{dist}(c, \partial Y_{k_n}) - \max_{w \in \partial X_{k_n}} |w - c|} \leq \frac{2 \max_{w \in \partial X_{k_n}} |w - c|}{\text{dist}(c, \partial Y_{k_n}) - \max_{w \in \partial X_{k_n}} |w - c|} \\
 &\leq \frac{2C_2(d) \text{dist}(c, \partial X_{k_n})}{C_1^{-1}(d) \max_{w \in \partial Y_{k_n}} |w - c| - C_2(d) \text{dist}(c, \partial X_{k_n})} = \frac{2C_2(d)}{\frac{\max_{w \in \partial Y_{k_n}} |w - c|}{C_1(d) \text{dist}(c, \partial X_{k_n})} - C_2(d)}.
 \end{aligned} \tag{3.10}$$

On the other hand, by Lemma 2.5, the modulus of  $Y_{k_n} \setminus \bar{X}_{k_n}$  satisfies

$$\begin{aligned}
 &\frac{1}{2\pi} \log \frac{\max_{w \in \partial Y_{k_n}} |w - c|}{\text{dist}(c, \partial X_{k_n})} \geq \text{mod}(Y_{k_n} \setminus \bar{X}_{k_n}) \\
 &\geq \frac{1}{d} \text{mod}(\mathbb{D}(c_{k_n}, \delta/2) \setminus \overline{\mathbb{D}(c_{k_n}, \epsilon_{k_n})}) = \frac{1}{2\pi d} \log \frac{\delta}{2\epsilon_{k_n}}.
 \end{aligned} \tag{3.11}$$

Note that  $\epsilon_{k_n} \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that the relative distance  $\Delta(\partial U_{k_n}, \partial U'_{k_n})$  of  $\partial U_{k_n}$  and  $\partial U'_{k_n}$  tends to zero as  $n$  tends to  $\infty$  by (3.10) and (3.11). This means that the peripheral circles of  $J_f$  are not uniformly relatively separated. The proof is completed.  $\square$

**3.5. Proofs of the main results.** We now give the proofs of the main results in the introduction by combining some propositions.

*Proof of Theorem 1.1.* By Propositions 3.6 and 3.8, the peripheral circles of carpet  $J_f$  are uniform quasicircles and uniformly relatively separated. According to Bonk [Bon, Corollary 1.2],  $J_f$  is quasimetrically equivalent to a round carpet.  $\square$

*Proof of Theorem 1.3.* The theorem follows immediately by Propositions 3.7, 3.8 and 3.9.  $\square$

*Proof of Theorem 1.4.* By Theorem 1.1, let  $g : J_f \rightarrow S$  be a quasimetric map sending  $J_f$  to a round carpet  $S$ . According to [Bon, Theorem 1.1], one can extend  $g : J_f \rightarrow S$  to a quasiconformal map from  $\widehat{\mathbb{C}}$  to itself. Since  $f$  is semi-hyperbolic, the corresponding Julia set  $J_f$  has measure zero by [Yin, Theorem 1.3]. It is well known that quasiconformal maps on the plane preserve the measure zero. So the round carpet  $S$  has measure zero as well. By the rigidity of Schottky sets (see [BKM, Theorem 1.1]), the quasimetric group  $QS(S)$  consists of the restriction of Möbius transformations.

Note that  $g$  induces a group isomorphism

$$g_* : QS(J_f) \rightarrow QS(S) \quad \text{with} \quad g_*(h) = g \circ h \circ g^{-1}.$$

We are left to show that  $QS(J_f)$  is *discrete*, i.e., there exists  $\delta > 0$  such that

$$\inf_{h \in QS(J_f) \setminus \{\text{id}_{J_f}\}} (\max_{z \in J_f} |h(z) - z|) \geq \delta.$$

If not, there exists a pairwise distinct sequence  $\{h_k\}_{k \geq 1} \subseteq QS(J_f)$  converging to  $\text{id}_{J_f}$ . Let  $C_1, C_2$  and  $C_3$  be three different peripheral circles of  $J_f$ . Then the Hausdorff distance between  $C_i$  and  $h_k(C_i)$  tends to zero as  $k$  tends to  $\infty$ . Since all  $h_k(C_i)$  are either disjoint or coincides for  $k \geq 1$  and  $i \in \{1, 2, 3\}$ , it follows that  $h_k(C_i) = C_i$  for sufficiently large  $k$ . This means, for sufficiently large  $k$ , the Möbius transformation  $g_*(h_k)$  fixes three disjoint round disks bounded by  $g(C_i)$ , where  $1 \leq i \leq 3$ .

By the rigidity of Möbius transformation, these  $g_*(h_k)$  must be the identity  $\text{id}_S$ . It follows that  $h_k = \text{id}_{J_f}$ , for sufficiently large  $k$ . This contradicts the choice of  $\{h_k\}_{k \geq 1}$ . The discreteness of  $QS(J_f)$  is proved.  $\square$

#### 4 An example of postcritically-infinite carpet Julia set

In this section, we will construct a carpet Julia set of a rational map such that it is quasimetrically equivalent to a round carpet. However, the rational map  $f$  is semi-hyperbolic and has an infinite critical orbit in  $J_f$ .

Let  $q : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$  be the *doubling map* defined by  $q(t) = 2t \bmod \mathbb{Z}$  and

$$l(t) := \begin{cases} 2t & \text{if } 0 \leq t < 1/2, \\ 2 - 2t & \text{if } 1/2 \leq t < 1. \end{cases} \quad (4.1)$$

be the length of the component  $(\mathbb{R}/\mathbb{Z}) \setminus \{t, 1 - t\}$  containing 0. Let  $T(t) = \min\{2t, 2 - 2t\}$  be the *tent map* on the interval  $[0, 1]$ . One can easily check that

$$T \circ l(t) = l \circ q(t) \quad (4.2)$$

for all  $t \in [0, 1]$ . Actually, the map  $l(t)$  is equal to  $T(t)$ . We use these notations here by following Tiozzo's paper [Ti, p. 24].

**Lemma 4.1.** *Let  $0 \leq \alpha \leq 1$  be a real number. Then  $\alpha$  is rational if and only if  $\alpha$  is (pre-)periodic under the iteration of the doubling map  $q$ .*

*Proof.* Obviously, this lemma holds for  $\alpha = 0$  or  $1$ . Hence we assume that  $0 < \alpha < 1$ . If  $\alpha$  is (pre-)periodic, then there exist two different integers  $k_1, k_2 \geq 0$  such that  $q^{o k_1}(\alpha) \equiv q^{o k_2}(\alpha) \bmod \mathbb{Z}$ . This means that there exists an integer  $k_3$  such that  $2^{k_1}\alpha = 2^{k_2}\alpha + k_3$ . Then  $\alpha = k_3/(2^{k_1} - 2^{k_2})$  is a rational number.

Conversely, we only need to prove that, if  $\alpha = m/n$  is a rational number with the simplest expression, where  $n$  is odd, then  $\alpha$  is periodic under  $q$ . Consider the restriction of  $q$  on the set  $S := \{0, 1/n, \dots, (n-1)/n\}$ :

$$h := q|_S : \frac{t}{n} \mapsto \frac{2t \bmod n}{n}.$$

We claim that  $h$  is injective. Indeed, if  $h(t_1/n) = h(t_2/n)$ , then  $2(t_1 - t_2) = kn$  holds for some integer  $k$ . Since  $n$  is odd, it follows that  $k$  is even and  $|t_1 - t_2| = |\frac{k}{2}| \cdot n \leq n - 1$ . This means that  $k = 0$  and  $t_1 = t_2$ . The finiteness of the cardinal number of  $S$  implies every element in  $S$  is pre-periodic under  $h$ . Then each element in  $S$  is periodic. Otherwise, there will be at least two elements which are mapped to a same element. This contradicts with that  $h$  is a injection. The proof is complete.  $\square$

In the following, based on the combinatorial theory of quadratic polynomials and renormalization theory, we shall construct a semi-hyperbolic McMullen map whose Julia set is quasimetrically equivalent to a round carpet.

**Theorem 4.2.** *There exists a suitable parameter  $\lambda > 0$  such that the McMullen map*

$$f_\lambda(z) = z^d + \lambda/z^d \quad (4.3)$$

*is semi-hyperbolic and the corresponding Julia set is quasimetrically equivalent to a round Sierpiński carpet, where  $d \geq 3$ .*

*Proof.* We divide the construction into three main steps as following.

**Step 1.** For a given irrational number  $\alpha \in (0, 1)$ , one can write it as an infinite binary sequence  $\alpha = 0.a_1a_2a_3\dots$  by Lemma 4.1, where  $a_i \in \{0, 1\}$ . Define a binary number

$$\theta := 0.0 \underbrace{1\dots 1}_{100} \underbrace{0\dots 0}_{b_1} \underbrace{1\dots 1}_{b_2} \underbrace{0\dots 0}_{b_3} \dots$$



with  $b_i = a_i + 1$  for  $i \geq 1$ . Then  $1 \leq b_i \leq 2$  and we have:

- The number  $\theta \in (0, 1/2)$  is irrational. If not, by Lemma 4.1, the number  $\theta$  will be eventually periodic under the iteration of the doubling map  $q$ . Then there exist  $m \geq 2$  and  $p \geq 2$  such that

$$q^{on}(\theta) = 0.\underbrace{1 \cdots 1}_{b_m} \underbrace{0 \cdots 0}_{b_{m+1}} \cdots \underbrace{0 \cdots 0}_{b_{m+p-1}},$$

where  $n = 101 + b_1 + \cdots + b_{m-1}$ . This means that the sequence  $b_m, b_{m+1}, \dots, b_{m+p-1}, b_{m+p}, \dots$  is periodic with period  $p$ . Therefore, the sequence  $a_m, a_{m+1}, \dots, a_{m+p-1}, a_{m+p}, \dots$  is also periodic with period  $p$  since  $a_i = b_i - 1$  for each  $i$ . Then  $\alpha = 0.a_1 \cdots a_{m-1} \overline{a_m a_{m+1} \cdots a_{m+p-1}}$  is a rational number by Lemma 4.1. This is a contradiction since  $\alpha$  is irrational.

- Define a rational number with the binary form

$$\theta' = 0.\overline{01 \cdots 1}.$$

Then  $0 < \theta' < \theta < 1/2$  and  $\theta', \theta$  are very close to  $1/2$ . We have

$$l(\theta') = 0.\underbrace{1 \cdots 1}_{99} \overline{01 \cdots 1} \text{ and } l(\theta) = 0.\underbrace{1 \cdots 1}_{100} \underbrace{0 \cdots 0}_{b_1} \underbrace{1 \cdots 1}_{b_2} \underbrace{0 \cdots 0}_{b_3} \cdots$$

For any  $n \geq 2$ , one can easily check that

$$0 < l(q(\theta')) < l(q^{on}(\theta)), l(q^{on}(\theta')) < l(\theta') < l(\theta). \quad (4.4)$$

- Define a set

$$\mathcal{R} := \{t \in \mathbb{R}/\mathbb{Z} : T^{on}(l(t)) \leq l(t) \text{ for all } n \geq 0\}. \quad (4.5)$$

By (4.2) and (4.4), we have  $\theta', \theta \in \mathcal{R}$ .

**Step 2.** Construct a quadratic polynomial  $P_c(z) = z^2 + c$  with the following properties:

(1) The critical orbit  $O_{P_c}^+(0) = \{P_c^n(0) : n \geq 0\}$  is contained in the Julia set of  $P_c$  and the cardinal number of  $O_{P_c}^+(0)$  is infinite.

(2) The critical point 0 is non-recurrent and the  $\omega$ -limit set of 0 does not contain the  $\beta$  fixed point. Recall that a  $\beta$  fixed point of a polynomial is the landing point of *dynamical external ray* with angle zero.

The set  $\mathcal{R}$  defined in (4.5) is exactly the set of all angles of parameter rays whose *prime-end impression* intersects the subset  $\mathbb{R} \cap M = [-2, 1/4]$  of the Mandelbrot set  $M$  (see [Ti, Proposition 8.4]). By [Za, Theorem 3.3], there exists a real number  $c := c(\theta) \in [-2, 1/4]$  in the boundary of the Mandelbrot set such that  $c$  is contained in the prime-end impression of the parameter rays  $R_M(\pm\theta)$  since  $\theta \in \mathcal{R}$ . Moreover, on the dynamical plane, the dynamical rays  $R_c(\pm\theta)$  land at the critical value  $c$  of  $P_c(z) = z^2 + c$ .

In fact, such  $c$  is unique. Otherwise, suppose that there exists another  $c' \neq c$ , such that  $c'$  is contained in the prime-end impression of the parameter rays  $R_M(\pm\theta)$ . By the density of hyperbolic parameters in  $\mathbb{R} \cap M$  (see [GS] and [Ly]), there is a real hyperbolic parameter  $\tilde{c}$  between  $c$  and  $c'$  with a pair of rational parameter rays landing at it. This means that  $c'$  and  $c$  cannot lie in the same prime-end impression of  $R_M(\pm\theta)$  at the same time, which is a contradiction.

Now we prove that the quadratic polynomial  $P_c$  is the map what we want to find. Again by [Ti, Proposition 8.4], the parameter rays  $R_M(\pm\theta')$  land at a parabolic parameter  $c_0 \in \mathbb{R}$  since  $\theta' \in \mathcal{R}$  is a rational number. These two rays together with their landing point  $R_M(\theta') \cup R_M(-\theta') \cup \{c_0\}$  bounds a *wake*  $W \ni \{-2\}$  with the following property: The quadratic map  $P_\xi(z) = z^2 + \xi$  has a repelling periodic point with exactly two dynamical rays  $R_\xi(\pm\theta')$  landing at if and only if

$\xi \in W$  (see [Mi2, Theorem 1.2]). By the construction in Step 1, we have  $0 < \theta' < \theta < 1/2$ . Then  $R_M(\pm\theta) \cup \{c\} \subset W$  and hence  $R_c(\pm\theta')$  land at a repelling periodic point of  $P_c$  on the real line. Also, the image  $R_c(\pm q(\theta'))$  of  $R_c(\pm\theta')$  land at some point on the real line.

Denote by  $H$  the simply connected domain bounding by the four dynamical rays  $R_c(\pm\theta')$  and  $R_c(\pm q(\theta'))$ . The two dynamical rays  $R_c(\theta)$  and  $R_c(0)$  are contained in different components of  $\mathbb{C} \setminus \bar{H}$ . Moreover, all the dynamical rays  $R_c(\pm q^n(\theta))$ , where  $n \geq 2$ , are contained in  $H$  by the definition of  $\mathcal{R}$  and  $\theta'$ . This means that the collection of their landing points  $\bigcup_{n \geq 2} P_c^{on}(c)$  are contained in  $H$ . Therefore, the critical value  $c$  (which is the landing point of  $R_c(\pm\theta)$ ) and the  $\beta$  fixed point of  $P_c$  are not contained in the  $\omega$ -limit set of the origin.

**Step 3.** Construct the semi-hyperbolic rational map  $f_\lambda$  whose Julia set is quasymmetrically equivalent to a round carpet. Consider the McMullen map  $f_\lambda(z) = z^d + \lambda/z^d$ , where  $\lambda \in \mathbb{C} \setminus \{0\}$  and  $d \geq 3$ . The *free* critical points of  $f_\lambda$  are  $2d$ -th unit roots of  $\lambda$ . They are either escaping to  $\infty$  or have bounded orbits at the same time. The *non-escaping locus* of  $f_\lambda$  is defined as

$$\Lambda_d := \{\lambda \in \mathbb{C} \setminus \{0\} : \text{The free critical orbits of } f_\lambda \text{ are not attracted by } \infty\}.$$

See left picture in Figure 4 for the case when  $d = 3$ .

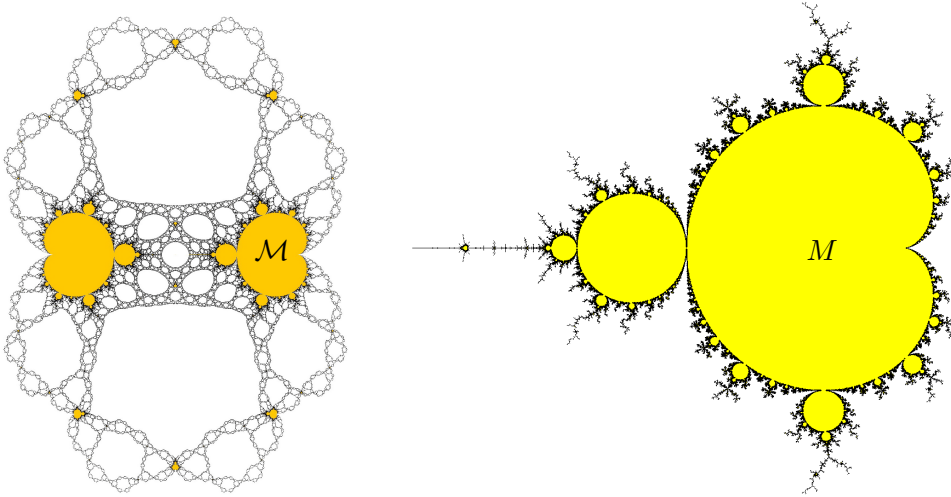


Figure 4: The non-escaping locus  $\Lambda_3$  of  $f_\lambda$  (the left picture) contains infinitely many homeomorphic copies of the Mandelbrot set (the right picture).

According to [Ste, Theorem 9], there exists exactly one copy  $\mathcal{M}$  of the Mandelbrot set of *order* one in  $\Lambda_d \cap \{\lambda \in \mathbb{C}^* : |\arg(\lambda)| < \pi/(d-1)\}$  (Note that there exists a semiconjugacy between  $f_\lambda$  and the rational map discussed in [Ste, Theorem 9]). The copy  $\mathcal{M}$  is symmetric with respect to the positive real axis. Moreover, there exists a homeomorphism  $\Phi : \mathcal{M} \rightarrow M$  such that, for every  $\lambda \in \mathbb{R}^+ \cap \mathcal{M} = \mathbb{R}^+ \cap \Lambda_d$ , there is a corresponding parameter  $\Phi(\lambda) \in [-2, 1/4]$  and the Julia set  $J_{f_\lambda}$  contains an embedded set  $\tilde{J}_{P_{\Phi(\lambda)}}$ , which is homeomorphic to the Julia set of the quadratic polynomial  $P_{\Phi(\lambda)}(z) = z^2 + \Phi(\lambda)$ . Moreover, the restriction of  $f_\lambda$  in a neighborhood of  $\tilde{J}_{P_{\Phi(\lambda)}}$  is quasiconformally conjugated to the restriction of  $P_{\Phi(\lambda)}$  in a neighborhood of  $J_{P_{\Phi(\lambda)}}$ .

Let  $\lambda_0 = \Phi^{-1}(c) \in \mathcal{M} \cap \mathbb{R}^+$ , where  $c = c(\theta) \in (-2, 1/4)$  is the real parameter on the boundary of the Mandelbrot set determined in Step 2. By the symmetry of McMullen maps, all  $2d$  free critical points of  $f_{\lambda_0}$  are non-recurrent and they have infinite forward orbits. This means that  $f_{\lambda_0}$  is semi-hyperbolic (and not sub-hyperbolic). Let  $B_\infty$  be the immediate attracting basin of  $\infty$  of  $f_{\lambda_0}$ . Then  $\tilde{J}_{P_{\Phi(\lambda_0)}} \cap B_\infty = \{z_{\lambda_0}\}$ , where  $z_{\lambda_0}$  is the image of the  $\beta$  fixed point of  $J_{P_c}$  under

the quasiconformal conjugacy stated above [QXY, Lemma 4.1]. Note that  $B_\infty$  is the unique periodic Fatou component of  $f_{\lambda_0}$ , it follows that the  $\omega$ -limit sets of the critical points of  $f_{\lambda_0}$  are disjoint with the periodic Fatou component of  $f_{\lambda_0}$  by the construction of  $P_c$ .

By [QXY, Lemma 4.4], the Julia set of  $f_{\lambda_0}$  is a Sierpiński carpet. By Theorem 1.3, the peripheral circles of  $J_{f_{\lambda_0}}$  are uniform quasicircles and uniformly relatively separated. By Bonk's criterion ([Bon, Corollary 1.2]), the Julia set of  $f_{\lambda_0}$  is quasimetrically equivalent to a round carpet, as required.  $\square$

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